C4 UK June 16 Kprime2

1. Use the binomial series to find the expansion of

$$\frac{1}{(2+5x)^3}, \qquad |x| < \frac{2}{5}$$

in ascending powers of x, up to and including the term in x^3.
Give each coefficient as a fraction in its simplest form. **(6)**

$$(2+5x)^{-3} = \left[2\left(1+\frac{5}{2}x\right)\right]^{-3}$$

$$= \frac{1}{8}\left(1+\frac{5}{2}x\right)^{-3}$$

$$= \frac{1}{8}\left(1 + (-3)\left(\frac{5}{2}x\right) + \frac{(-3)(-4)}{2!}\left(\frac{5}{2}x\right)^2 + \frac{(-3)(-4)(-5)}{3!}\left(\frac{5}{2}x\right)^3\right)$$

$$= \frac{1}{8}\left(1 - \frac{15}{2}x + \frac{75}{2}x^2 - \frac{625}{4}x^3\right) \ldots$$

$$= \frac{1}{8} - \frac{15}{16}x + \frac{75}{16}x^2 - \frac{625}{32}x^3 \ldots$$

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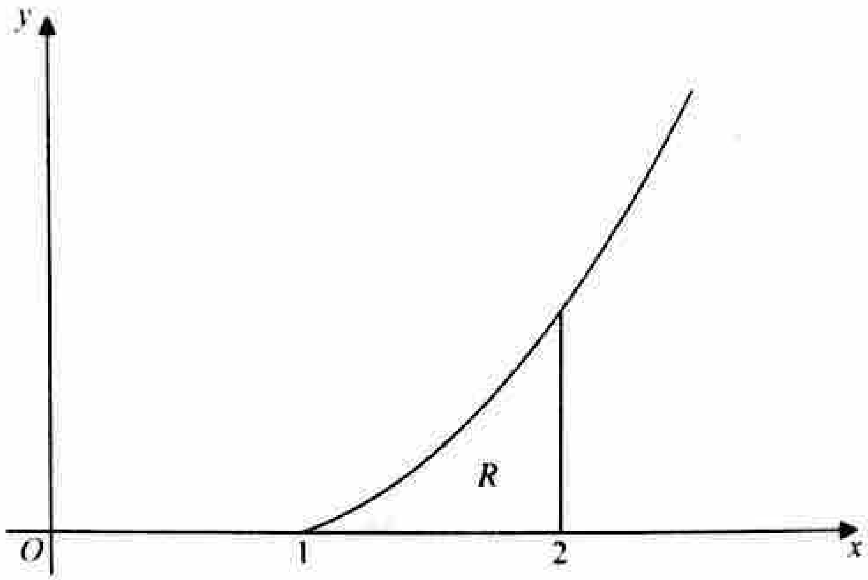

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = x^2 \ln x, \quad x \geqslant 1$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the x-axis and the line $x = 2$

The table below shows corresponding values of x and y for $y = x^2 \ln x$

x	1	1.2	1.4	1.6	1.8	2
y	0	0.2625	0.6595	1.2032	1.9044	2.7726

(a) Complete the table above, giving the missing value of y to 4 decimal places.

(1)

(b) Use the trapezium rule with all the values of y in the completed table to obtain an estimate for the area of R, giving your answer to 3 decimal places.

(3)

(c) Use integration to find the exact value for the area of R.

(5)

(b) Area $\approx \frac{1}{2}\left(\frac{1}{5}\right)\left[0 + 2(0.2625 + 0.6595 + 1.2032 + 1.9044) + 2.7726\right]$

$= 1.083$ (3dp)

Let $u = \ln x$ $\quad v' = $

$u' = \frac{1}{x}$

$v' = x^2 \quad v = \frac{1}{3}x^3$

(C) $\int_1^2 x^2 \ln x \, dx$

$= \left[\frac{x^3 \ln x}{3}\right]_1^2 - \frac{1}{3}\int_1^2 \frac{x^3}{x} \, dx$

$= \frac{8}{3}\ln 2 - \frac{1}{3}\int_1^2 x^2 \, dx$

$= \frac{8}{3}\ln 2 - \frac{1}{3}\left[\frac{1}{3}x^3\right]_1^2$

Area R $= \frac{8}{3}\ln 2 - \frac{7}{9}$

3. The curve C has equation

$$2x^2y + 2x + 4y - \cos(\pi y) = 17$$

(a) Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

(5)

The point P with coordinates $\left(3, \frac{1}{2}\right)$ lies on C.

The normal to C at P meets the x-axis at the point A.

(b) Find the x coordinate of A, giving your answer in the form $\frac{a\pi + b}{c\pi + d}$, where a, b, c and d are integers to be determined.

(4)

3(a) $\frac{\partial}{\partial x}\left(2x^2y + 2x + 4y - \cos \pi y\right) = \frac{\partial}{\partial x}(17)$

$\therefore\ 4xy + 2x^2\frac{\partial y}{\partial x} + 2 + 4\frac{\partial y}{\partial x} + \pi \sin \pi y \frac{\partial y}{\partial x} = 0$

$\cancel{\Rightarrow (2x^2+4)\frac{\partial y}{\partial x} = -4xy - \pi \sin \pi y - 2}$

$\Rightarrow (2x^2 + \pi \sin \pi y + 4)\frac{\partial y}{\partial x} = -4xy - 2$

$\therefore\ \frac{\partial y}{\partial x} = \frac{-4xy - 2}{2x^2 + \pi \sin \pi y + 4}$

Question 3 continued

(b) @ P, $\frac{dy}{dx} = \frac{-8}{22+\pi}$

Gradient at normal $= \frac{\pi+22}{8}$

$y - y_1 = m(x - x_1)$

$\therefore y - \frac{1}{2} = \frac{\pi+22}{8}(x-3)$

$y = 0$ @ A

$\Rightarrow -\frac{1}{2} = \left(\frac{\pi+22}{8}\right)(x-3)$

$\therefore \frac{-4}{\pi+22} = x-3$

$\therefore x = 3 - \frac{4}{\pi+22}$

$= \frac{3\pi+66}{\pi+22} - \frac{4}{\pi+22}$

$x_A = \frac{3\pi+62}{\pi+22}$

$a = 3$
$b = 62$
$c = 1$
$d = 22$

Turn over

P 4 6 7 1 8 A 0 9 3 2

4. The rate of decay of the mass of a particular substance is modelled by the differential equation

$$\frac{dx}{dt} = -\frac{5}{2}x, \qquad t \geqslant 0$$

where x is the mass of the substance measured in grams and t is the time measured in days.

Given that $x = 60$ when $t = 0$,

(a) solve the differential equation, giving x in terms of t. You should show all steps in your working and give your answer in its simplest form.

(4)

(b) Find the time taken for the mass of the substance to decay from 60 grams to 20 grams. Give your answer to the nearest minute.

(3)

4(a) $\frac{\partial x}{\partial t} = -\frac{5}{2}x$

$\therefore \frac{1}{x}\frac{\partial x}{\partial t} = -\frac{5}{2}$

$\therefore \int \frac{1}{x}\frac{\partial x}{\partial t}\,\partial t = \int -\frac{5}{2}\,\partial t$

$\Rightarrow \int \frac{1}{x}\,\partial x = -\frac{5}{2}t + k$

$\therefore \ln x = -\frac{5}{2}t + c$

$\left.\begin{array}{l} x = 60 \\ t = 0 \end{array}\right\} \therefore \ln 60 = c$

$\Rightarrow \ln x = -\frac{5}{2}t + \ln 60$

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Question 4 continued

$\therefore \quad e^{\ln x} = e^{-\frac{5}{2}t + \ln 60}$

$\therefore \quad x = e^{-\frac{5}{2}t} \times e^{\ln 60}$

$\therefore \quad x = 60e^{-\frac{5}{2}t}$

(b) $x = 20 \Rightarrow 20 = 60e^{-5/2\, t}$

$\therefore \quad \frac{1}{3} = e^{-\frac{5}{2}t}$

$\therefore \quad -\frac{5}{2}t = \ln\left(\frac{1}{3}\right) = -\ln(3)$

$\therefore \quad t = \frac{2}{5}\ln 3$

$\Rightarrow t = 0.43944\ldots$ DAYS

$\times 24 \times 60 \Rightarrow$

$\therefore$ Time taken is 633 minutes

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Turn over

5.

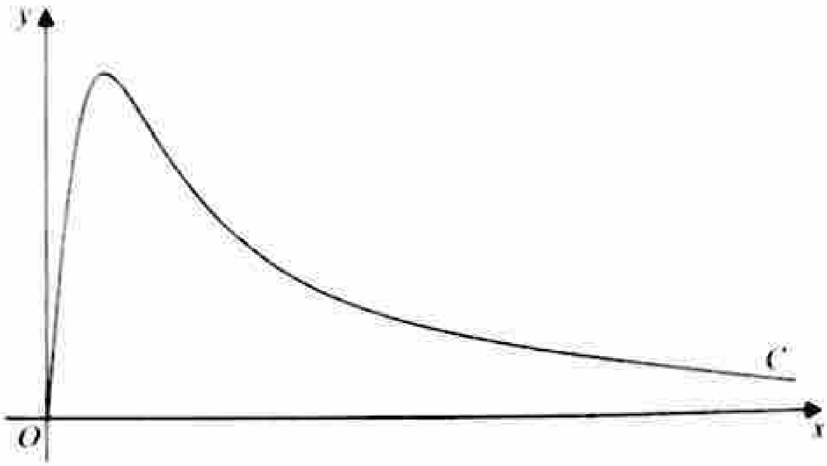

Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 4\tan t, \qquad y = 5\sqrt{3}\sin 2t, \qquad 0 \leqslant t < \frac{\pi}{2}$$

The point P lies on C and has coordinates $\left(4\sqrt{3}, \frac{15}{2}\right)$.

(a) Find the exact value of $\frac{dy}{dx}$ at the point P.

Give your answer as a simplified surd.

(4)

The point Q lies on the curve C, where $\frac{dy}{dx} = 0$

(b) Find the exact coordinates of the point Q.

(2)

5(a). $\frac{\partial y}{\partial t} = 10\sqrt{3}\cos 2t$ $\quad$ $x = 4\sqrt{3} \Rightarrow t = \pi/3$

$y = \frac{15}{2} \Rightarrow t = \left\{\frac{\pi}{6}, \frac{\pi}{3}\right\}$

$\frac{\partial x}{\partial t} = 4\sec^2 t$ $\quad$ $0 \leq 2t < \pi$ $\quad$ t

$\therefore \frac{\partial y}{\partial x} = \frac{10\sqrt{3}\cos 2t}{4\sec^2 t} \times \frac{\cos^2 t}{\cos^2 t}$

$= \frac{10\sqrt{3}\cos^2 t\cos 2t}{4}$

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Question 5 continued

$t = \frac{\pi}{3}$

@ P, $\frac{\partial y}{\partial x} = \frac{-5\sqrt{3}}{16}$

(b) $\frac{\partial x}{\partial x} = \frac{10\sqrt{3}\cos^2 t \cos 2t}{4} = 0$

$\therefore \cos^2 t \cos 2t = 0$

S | A
T | C

$\cos t = 0$

$\Rightarrow t = \pi/2$ X reject

$\cos 2t = 0 \Rightarrow 2t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{4}$

$\therefore x = 4 \tan \pi/4 = 4$

$y = 5\sqrt{3} \sin\left(\frac{\pi}{2}\right)$

$= 5\sqrt{3}$

$\therefore Q(4, 5\sqrt{3})$

6. (i) Given that $y > 0$, find

$$\int \frac{3y-4}{y(3y+2)}\,dy$$

(6)

(ii) (a) Use the substitution $x = 4\sin^2\theta$ to show that

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx = \lambda \int_0^{\frac{\pi}{3}} \sin^2\theta\,d\theta$$

where λ is a constant to be determined.

(5)

(b) Hence use integration to find

$$\int_0^3 \sqrt{\left(\frac{x}{4-x}\right)}\,dx$$

giving your answer in the form $a\pi + b$, where a and b are exact constants.

(4)

$$6(i)\quad \frac{3y-4}{y(3y+2)} = \frac{A}{y} + \frac{B}{3y+2}$$

$$\therefore 3y-4 = A(3y+2) + By$$

$$y=0 \Rightarrow -4 = 2A \Rightarrow A = -2$$

$$y = -\frac{2}{3} \Rightarrow -6 = -\frac{2}{3}B \Rightarrow B = 9$$

$$\therefore \int \frac{3y-4}{y(3y+2)}\,dy = \int -\frac{2}{y} + \frac{9}{3y+2}\,dy$$

$$= -2\ln y + 3\ln(3y+2) + C$$

Question 6 continued

(ii) (a) $x = 4\sin^2\theta$ $\qquad \frac{\partial x}{\partial \theta} = 8\sin\theta\cos\theta$

Limits: $\qquad \partial x = 8\sin\theta\cos\theta\,\partial\theta$

$x = 3 \Rightarrow 3 = 4\sin^2\theta \Rightarrow \theta = \arcsin\left(\sqrt{\frac{3}{4}}\right) = \pi/3$

$x = 0 \Rightarrow 0 = 4\sin^2\theta \Rightarrow \theta = \arcsin 0 = 0$

$$\int_0^3 \sqrt{\frac{x}{4-x}}\,\partial x = \int_0^{\pi/3} \sqrt{\frac{4\sin^2\theta}{4-4\sin^2\theta}}\,8\sin\theta\cos\theta\,\partial\theta$$

$$= \int_0^{\pi/3} \sqrt{\frac{4\sin^2\theta}{4\cos^2\theta}}\,8\sin\theta\cos\theta$$

$$= \int_0^{\pi/3} \sqrt{\tan^2\theta}\,8\sin\theta\cos\theta\,\partial\theta$$

$$= 8\int_0^{\pi/3} \tan\theta\sin\theta\cos\theta\,\partial\theta$$

$$= 8\int_0^{\pi/3} \frac{\sin^2\theta\cos\theta}{\cos\theta}\,\partial\theta$$

$$= \int_0^{\pi/3} \sin^2\theta\,\partial\theta$$

Question 6 continued

$\cancel{\cos 2\theta = 2\cos^2\theta - 1}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cancel{\therefore \sin^2\theta = \cos 2\theta - 1}$

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

(b) $8\int_0^{\pi/3} \sin^2\theta \, d\theta$

$= 4\int_0^{\pi/3} 1 - \cos 2\theta \, d\theta$

$= 4\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/3}$

$= 4\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$

$= \frac{4}{3}\pi - \sqrt{3}$

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7. (a) Find

$$\int (2x-1)^{\frac{3}{2}}\,dx$$

giving your answer in its simplest form. $\int (ax+b)^n \, dx = \frac{1}{a(n+1)}(ax+b)^{n+1}$ **(2)**

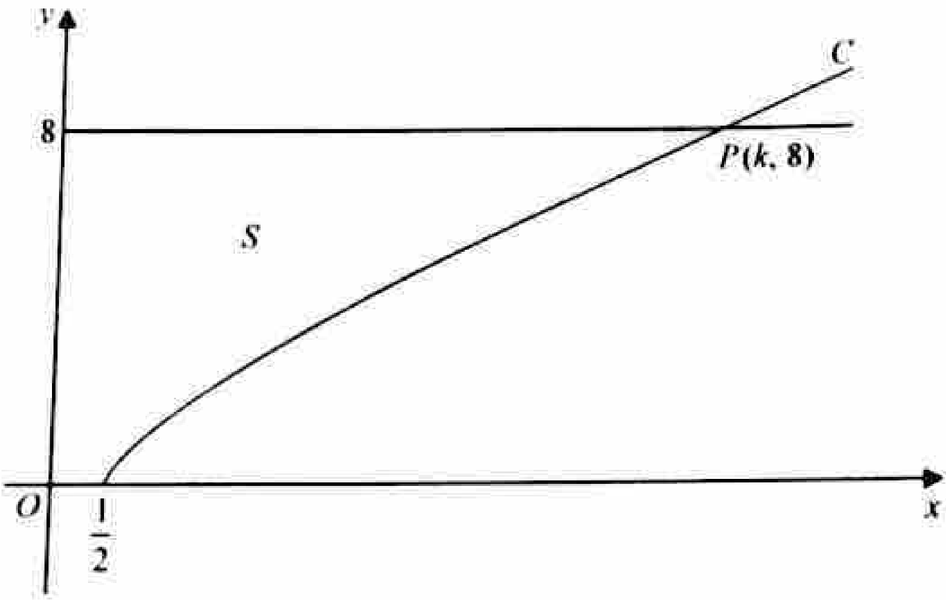


Figure 3

Figure 3 shows a sketch of part of the curve C with equation

$$y = (2x-1)^{\frac{3}{4}}, \qquad x \geqslant \frac{1}{2}$$

The curve C cuts the line $y = 8$ at the point P with coordinates $(k, 8)$, where k is a constant.

(b) Find the value of k.

(2)

The finite region S, shown shaded in Figure 3, is bounded by the curve C, the x-axis, the y-axis and the line $y = 8$. This region is rotated through 2π radians about the x-axis to form a solid of revolution.

(c) Find the exact value of the volume of the solid generated.

(4)

7(a). $\int (2x-1)^{3/2} \, dx = \frac{1}{5}(2x-1)^{5/2} + C$

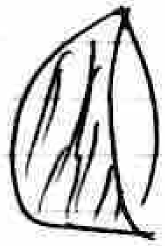
Question 7 continued

(b) $y = 8 \Rightarrow \quad 8 = (2k-1)^{3/4}$

$8^{4/3} = 2k - 1$

$\therefore 16 = 2k - 1 \Rightarrow \quad k = \frac{17}{2}$

(c)



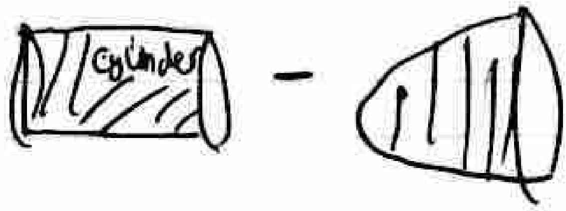
$V = \pi \int y^2 \, dx$

$V = \pi \int_{1/2}^{17/2} (2x-1)^{3/2} \, dx$

$= \pi \left[\frac{1}{5}(2x-1)^{5/2} \right]_{1/2}^{17/2}$

$= \frac{1024}{5}\pi$

Volume = $\pi r^2 h$



$\frac{17}{2} = h$

$r = 8$

$= \pi r^2 h - \frac{1024}{5}\pi$

$= 544\pi - \frac{1024}{5}\pi$

$V = \frac{1696}{5}\pi$

Q7

(Total 8 marks)

8. With respect to a fixed origin O, the line l_1 is given by the equation

$$\mathbf{r} = \begin{pmatrix} 8 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$$

where μ is a scalar parameter.

The point A lies on l_1 where $\mu = 1$

(a) Find the coordinates of A.
(1)

The point P has position vector $\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$.

The line l_2 passes through the point P and is parallel to the line l_1

(b) Write down a vector equation for the line l_2
(2)

(c) Find the exact value of the distance AP.
Give your answer in the form $k\sqrt{2}$, where k is a constant to be determined.
(2)

The acute angle between AP and l_2 is θ.

(d) Find the value of $\cos\theta$
(3)

A point E lies on the line l_2
Given that $AP = PE$,

(e) find the area of triangle APE.
(2)

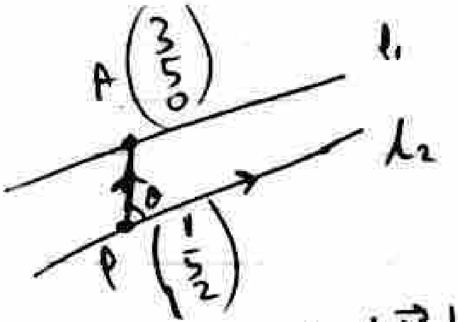
(f) find the coordinates of the two possible positions of E.
(5)

8(a). $\begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$ ✓✓

Question 8 continued

(b) $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix}$

(c)



$\overrightarrow{PA} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

$\therefore |\overrightarrow{PA}| = \sqrt{2^2+2^2}$

$|AP| = 2\sqrt{2}$

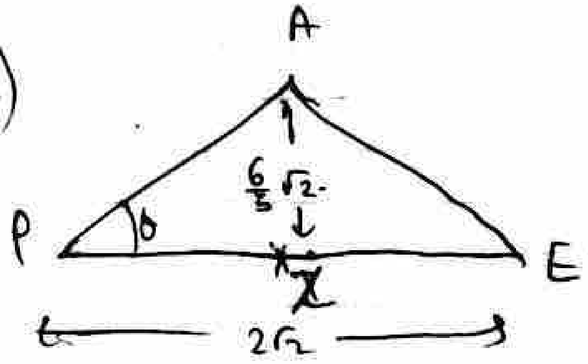
(d) $\cos\theta = \dfrac{\left| \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -5 \\ 4 \\ 3 \end{pmatrix} \right|}{2\sqrt{2} \times \sqrt{5^2+4^2+3^2}}$

$= \dfrac{16}{20} = \dfrac{4}{5}$

$\cos\theta = \dfrac{4}{5}$

Question 8 continued

(e)



$\sin\theta = \frac{3}{5}$

$|AX| = |AP|\sin\theta = 2\sqrt{2} \times \frac{3}{5}$

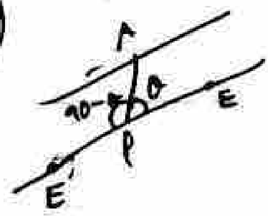
$|AX| = \frac{6}{5}\sqrt{2}$

$\therefore$ Area $= \frac{1}{2} \times$ base $\times$ height

$\Rightarrow$ Area $= \frac{1}{2} \times 2\sqrt{2} \times \frac{6}{5}\sqrt{2}$

Area $= \frac{12}{5}$

(f)



Let E be $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}$

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}1-5\lambda\\5+4\lambda\\2+3\lambda\end{pmatrix}$$

$\alpha = 1-5\lambda$

$\beta = 5+4\lambda$

$\gamma = 2+3\lambda$

Question 8 continued

$$\vec{PE} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha - 1 \\ \beta - 5 \\ \gamma - 2 \end{pmatrix}$$

$$|PE| = |AP|$$

$$\therefore \sqrt{(\alpha-1)^2 + (\beta-5)^2 + (\gamma-2)^2} = 2\sqrt{2}$$

$$\Rightarrow (\alpha-1)^2 + (\beta-5)^2 + (\gamma-2)^2 = 8$$

$$\therefore (-5\lambda)^2 + (4\lambda)^2 + (3\lambda)^2 = 8$$

$$50\lambda^2 = 8$$

$$\lambda^2 = \frac{4}{25}$$

$$\therefore \lambda = \pm \frac{2}{5}$$

$$\therefore \lambda = \frac{2}{5} \Rightarrow E = \begin{pmatrix} -1 \\ 33/5 \\ 16/5 \end{pmatrix} \quad \lambda = -\frac{2}{5} \Rightarrow E = \begin{pmatrix} 3 \\ 17/5 \\ 4/5 \end{pmatrix}$$

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